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Orbifolds as Melvin geometry

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Abstract: In this paper we explicitly show that the various non-compact abelian orbifolds are realized as special limits of parameters in type-II (NSNS) Melvin background and its higher-dimensional generalizations. As a result the supersymmetric ALE spaces (A-type $\mathbb{C}^2/\mathbb{Z}_N$) and non-supersymmetric orbifolds in type-II and type-0 theory are all connected with each other by the exactly marginal deformation. Our results provide new examples of the duality between type-II and type-0 string theory. We also discuss the decay of unstable backgrounds in this model which include closed string tachyons.

Keywords: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

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Contents

1. Introduction

The Melvin-type magnetic background [1]–[5] in string theory [6]–[9] has many interesting aspects. This background has the structure of Kaluza-Klein theory in the sigma model description. One of intriguing aspects is that this model in superstring gives string theoretic examples of supersymmetry breaking which can be exactly solvable. This model includes three parameters (two magnetic parameters and one compactification radius), and in most parts of the parameter space this non-supersymmetric model includes closed string tachyons. In particular at the special value of the magnetic fields with the small radius limit in type-II theory the system is equivalent to type-0 string theory (see [10]–[15] and references there in). In this way type-0 string theory can be realized as a non-supersymmetric background in type-II superstring. Recently, this background has also been applied to M-theory (so called F7-brane [16, 17, 18]), which is properly said as the "9-11" flip of the above mentioned NSNS-Melvin background. This leads to the conjectured interpretation of type-0 theory as the $\mathbf{S}^1$ compactification of M-theory with the anti-periodic boundary condition for all fermions [12]. These facts may hopefully be useful for the closed string tachyon condensation (for recent discussions see e.g. [19, 20, 21]).

Another aspect which we would also like to stress in this paper is the relation to various non-compact orbifolds with a fixed point [22]. In particular, by constructing a higher-dimensional generalization of the NSNS-Melvin backgrounds we find the supersymmetric solvable string vacua which include all of (A-type) ALE orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ (see e.g. [23, 24, 25]) as the small radius limits with various fractional values of the magnetic parameters. We show this fact explicitly by proving that under such limits the one-loop partition functions reduce to those of orbifolds. For irrational values of the magnetic parameters we encounter unfamiliar string vacua which preserve half of thirty two supersymmetries and describe an "irrationally orbifolded" ten-dimensional non-compact space. This model may also be interpreted as a large-N limit of the orbifold $\mathbb{C}^2/\mathbb{Z}_N$. The higher-dimensional Melvin backgrounds we use can also be embedded in M-theory by "9-11" flip and these models include the F5-brane [16, 26].

Furthermore, we see that various two- or four-dimensional non-supersymmetric orbifolds in type-II and type-0 string can be also obtained as corners of the moduli space (see figure 2) of Melvin geometry in the same way. This can be regarded as new examples of the interpolation between the supersymmetric string theories and the non-supersymmetric ones. For example, one can see that the various non-supersymmetric orbifolds including the examples in [20] can be moved into the supersymmetric ones by marginal deformations. Such facts are useful for investigating various decay processes in the unstable closed string backgrounds as we will see later.

The plan of this paper is as follows. In section 2 we review some basic facts of the Melvin background as a solvable string model. After that, we compute its partition function of this model in the small radius limit, and see that the result is equivalent to two-dimensional orbifolds $\mathbb{C}/\mathbb{Z}_N$ in type-II and type-0 string if the magnetic parameters takes rational values. In section 3 we generalize the results into higher-dimensional Melvin backgrounds. In the same limit as before we obtain ALE orbifolds and various non-supersymmetric orbifolds. In section 4 we summarize the obtained results and discuss the closed string tachyon.

After writing this paper we noticed the preprint [27] on the net which discusses the closed string tachyon in the Melvin background. We also noted the paper [28], which has partial overlaps, after the present paper appears on the arXiv.

2. Two-dimensional orbifolds from Melvin background

Let us first briefly review the exactly solvable models of type-II superstring in the NS-NS Melvin backgrounds [8, 9]. See also [6, 7] for such models in bosonic string theory.

The target spaces of these models have the structure of Kaluza-Klein theory and they have the topology $\mathrm{M}_3 \times \mathbb{R}^{1,6}$. The three-dimensional manifold M_3 is given by $\mathbf{S}^1$ fibration over $\mathbb{R}^2$. We write the coordinates of $\mathbb{R}^2$ and $\mathbf{S}^1$ by (ρ, φ) (polar coordinate) and y (with radius R), respectively. This non-trivial fibration is due to two Kaluza-Klein (K.K.) gauge fields A_φ and B_φ (see eq. (2.1)), which originate from K.K. reduction of metric $G_{\varphi y}$ and B-field $B_{\varphi y}$, respectively.

The explicit metric and other NSNS fields before the Kaluza-Klein reduction are given as follows (we neglect the trivial flat part $\mathbb{R}^{1,6}$)

$$
\begin{aligned}
ds^2 &= d\rho^2 + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)} d\varphi^2 + \frac{1+q^2\rho^2}{1+\beta^2\rho^2}(dy + A_\varphi d\varphi)^2 \,, \\
A_\varphi &= \frac{q\rho^2}{1+q^2\rho^2} \,, \qquad B_{\varphi y} \equiv B_\varphi = -\frac{\beta\rho^2}{1+\beta^2\rho^2} \,, \qquad e^{2(\phi-\phi_0)} = \frac{1}{1+\beta^2\rho^2} \,, \qquad (2.1)
\end{aligned}
$$

where q, β are the parameters which are proportional to the strength of two gauge fields A_φ, B_φ and ϕ_0 is the constant value of the dilaton ϕ at $\rho = 0$.

It would be useful to note that if $\beta = 0$, then we get the locally flat metric

$$ds^2 = d\rho^2 + dy^2 + \rho^2 (d\varphi + q dy)^2 \,. \tag{2.2}$$

However, this background is globally non trivial because the angle φ is compactified such that its period is 2π. For example, its geodesics lines $\varphi + qy$ =constant are spiral and do not return to the same point for irrational qR if one goes around the circle $\mathbf{S}^1$ finite times. This fact shows the crucial difference between rational qR and irrational qR in physical arguments as we will see. This difference can also be seen explicitly in the D-brane spectrum in the NS-NS Melvin backgrounds [29].

At first sight the two-dimensional sigma model on the above curved background (2.1) for general q, β does not seem to be tractable. However, with appropriate T-duality transformations one can solve[1] this sigma model in terms of free fields [7, 8].

In order to see the detailed mass spectrums and the supersymmetry breaking in these models let us compute the one-loop partition function. This was computed in the Green-Schwarz formulation [8, 9] and we will transform its expression into that in the NS-R formulation.

On the world-sheet in the light-cone Green-Schwarz formulation, there are eight (real) bosonic fields ρ, φ, Y, X_i $(i = 2, 3, \dots, 6)$ and eight left-moving and right-moving fermionic fields S^a_L, S^a_R $(a = 1, 2, \dots, 8)$. The fermionic fields are divided into two groups $S^r_{L,R}$ and $\bar{S}^r_{L,R}$ $(r = 1, 2, 3, 4)$ according to U(1)-charge[2] $\hat{J}_{L,R} = 1/2$ and $-1/2$, where we defined U(1)-charge such that $X = \rho e^{i\varphi}$ and $\bar{X} = \rho e^{-i\varphi}$ also have the charge $\hat{J}_L = \hat{J}_R = 1$ and -1 as in [7, 9].

Now let us see the calculation of the partition function. Introducing the auxiliary vector field $V, \bar{V}$, we rewrite the world-sheet action as follows [8]

$$\begin{aligned}
S = &\frac{1}{\pi\alpha'} \int (d\sigma)^2 \Bigg[\bar{\partial}\rho\partial\rho + \left(1 + \beta^2\rho^2\right) V\bar{V} + V\left(\bar{\partial}Y + \beta\rho^2\bar{\partial}(\varphi + qY) + \frac{i\beta}{2}\bar{S}^r_R S^r_R\right) - \\
&\qquad - \bar{V}\left(\partial Y - \beta\rho^2\partial(\varphi + qY) + \frac{i\beta}{2}\bar{S}^r_L S^r_L\right) + \rho^2\partial(\varphi + qY)\bar{\partial}(\varphi + qY)\Bigg] \\
= &\frac{1}{\pi\alpha'} \int (d\sigma)^2 \Bigg[(\partial + i\beta V + iq\partial Y) X \left(\bar{\partial} - i\beta\bar{V} - iq\bar{\partial}Y\right)\bar{X} + \\
&\qquad + \bar{S}^r_R\left(\partial + \frac{i\beta}{2}V + \frac{iq}{2}\partial Y\right) S^r_R + \bar{S}^r_L\left(\bar{\partial} - \frac{i\beta}{2}\bar{V} - \frac{iq}{2}\bar{\partial}Y\right) S^r_L + \\
&\qquad + V\bar{V} - \bar{V}\partial Y + V\bar{\partial}Y\Bigg] \,.
\end{aligned} \tag{2.3}$$

Here we abbreviate the bosonic parts which come from the trivial directions $\mathbb{R}^{1,6}$. Note

[1] The background eq. (2.1) satisfies the equation of motion even if we take α' corrections into account [30].

[2] Operators $\hat{J}_{L,R}$ are angular momentum operators in the $(X'', \ \bar{X}'')$ plane.

that if we neglect the fermionic fields, then we can obtain the (bosonic) sigma model for the Melvin background (2.1) after we integrate out the auxiliary field $V,\ \bar{V}$.[3]

Next we would like to integrate out the field Y. Then the equation of motion (2.3) for Y shows $\bar{\partial}V - \partial\bar{V} = 0$ if we use also the equations of motion for $X,\ \bar{X}$. Therefore we can write V as

$$V = C + \partial\tilde{Y}\,, \qquad \bar{V} = \bar{C} + \bar{\partial}\tilde{Y}\,, \tag{2.4}$$

where C is the constant part; $\tilde{Y}$ is a bosonic field which has no zero-modes. Finally we obtain (we show only bosonic parts)

$$\begin{aligned} S = \frac{1}{\pi\alpha'}\int (d\sigma)^2 \Big[& C\bar{C} - \bar{C}\partial Y + C\bar{\partial}Y + \partial\tilde{Y}\bar{\partial}\tilde{Y} + \\ & + \left(\partial + i\beta C + i\beta\partial\tilde{Y} + iq\partial Y\right) X \cdot \left(\bar{\partial} - i\beta\bar{C} - i\beta\bar{\partial}\tilde{Y} - iq\bar{\partial}Y\right)\bar{X}\Big]\,. \end{aligned} \tag{2.5}$$

This shows that $X,\ S_R^r$ can also be written in terms of free field $X'',\ S_R^{\prime\prime r}$ such that

$$\begin{aligned} X(z,\bar{z}) &= e^{-i\beta(Cz+\bar{C}\bar{z})-i\beta\tilde{Y}-iqY} X''(z,\bar{z})\,, \\ S_R^r(z,\bar{z}) &= e^{-i\frac{\beta}{2}(Cz+\bar{C}\bar{z})-i\frac{\beta}{2}\tilde{Y}-i\frac{q}{2}Y} S_R^{\prime\prime r}(z,\bar{z})\,. \end{aligned} \tag{2.6}$$

Note also that the terms $\frac{1}{\pi\alpha'}\int(d\sigma)^2(-\bar{C}\partial Y + c.c.)$ involve only zero-mode parts of Y. Since the field Y has the periodicity $Y \sim Y + 2\pi R$, its zero-mode part is quantized in terms of winding numbers $w, w' \in \mathbb{Z}$ as follows

$$Y(\sigma_1,\sigma_2) = \sigma_1 w R + \sigma_2\left(w' - w\tau_1\right) R/\tau_2\,, \tag{2.7}$$

where $(\sigma_1,\sigma_2) \sim (\sigma_1+2\pi,\sigma_2) \sim (\sigma_1+2\pi\tau_1,\sigma_2+2\pi\tau_2)$ represents the coordinate of the torus.

Then it is easy to perform the path-integral[4] and the result is as follows [8]

$$\begin{aligned} Z(R,q,\beta) = & (2\pi)^{-7} V_7 R\left(\alpha'\right)^{-5} \int \frac{(d\tau)^2}{(\tau_2)^6} \times \\ & \times \int (dC)^2 \sum_{w,w'\in\mathbb{Z}} \frac{\left|\theta_1\left(\frac{\chi}{2}|\tau\right)\right|^8}{|\eta(\tau)|^{18}|\theta_1(\chi|\tau)|^2} \times \\ & \times \exp\left[-\frac{\pi}{\alpha'\tau_2}\left(4C\bar{C} - 2\bar{C}R\left(w' - w\tau\right) + 2CR\left(w' - w\bar{\tau}\right)\right)\right], \end{aligned} \tag{2.8}$$

where we have defined

$$\chi = 2\beta C + qR\left(w' - \tau w\right), \qquad \bar{\chi} = 2\beta\bar{C} + qR\left(w' - \bar{\tau}w\right). \tag{2.9}$$

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[3]For the related analysis of the curved backgrounds in Green-Schwarz formalism (light-cone gauge) see [31]. In principle it is possible for the world-sheet action in the formalism to include other terms higher than quartic in the fermions. However, these specific models are expected to have no higher terms since the free field representation is possible as shown by the T-duality. We thank A. A. Tseytlin for showing us this observation.

[4]Here we have redefined $C, \bar{C}$ such that $C \to i\bar{C}/\tau_2,\ \ \bar{C} \to -iC/\tau_2$.

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The last exponential factor comes from zero modes of Y. The theta-function terms originate from the following path-integral of non-zero modes

$$\frac{\det'(\partial-\bar{\chi}/(2\tau_2))\det'\left(\bar{\partial}-\chi/(2\tau_2)\right)}{\det'(\partial)\det'(\bar{\partial})} = \prod_{(n,n')\neq(0,0)} \frac{(n'-\tau n+\chi)\,(n'-\bar{\tau}n+\bar{\chi})}{(n'-\tau n)(n'-\bar{\tau}n)}$$
$$= e^{\pi(\chi-\bar{\chi})^2/2\tau_2}\left|\frac{\theta_1(\chi|\tau)}{\chi\theta_1'(0|\tau)}\right|^2 . \tag{2.10}$$

In this way we can obtain one-loop partition function in the Green-Schwarz formulation. It is easy to check its modular invariance using theta-function formulas eq. (A.2).

Now one may ask how to interpret the above result in the NS-R formulation. If one uses the Jacobi identity eq. (A.4)

$$\theta_3(0|\tau)^3\theta_3(\chi|\tau)-\theta_2(0|\tau)^3\theta_2(\chi|\tau)-\theta_4(0|\tau)^3\theta_4(\chi|\tau)=2\theta_1\left(\frac{\chi}{2}\middle|\,\tau\right)^4 , \tag{2.11}$$

then this explicitly represents the path-integral in the NS-R formulation with type-II GSO-projection. Note that the above partition function does not vanish in general and this implies the spacetime supersymmetry breaking. This physical result can also be understood in supergravity. To make matters simple let us assume $\beta=0$. If one goes around the circle $\mathbf{S}^1$, then all of spin 1/2 fermions receive the phase factor $e^{\pm i\pi qR}$ (see eq. (2.6)). Since this is not equal to one in general, there are no Killing spinors [13]. Only for $qR\in 2\mathbb{Z}$ the supersymmetry is preserved and this fact can be easily seen also from the periodicity (2.15).

Next we would like to relate the previous result to the mass spectrum in the operator formulation. Let us assume that the theta-functions are all expanded such that each term is an eigen state of the angular momentum operators $\hat{J}_R,\hat{J}_L$. Here we have defined the charge $\hat{J}_R,\hat{J}_L$ of the term such that it includes the factor $e^{2\pi i\chi\hat{J}_R+2\pi i\bar{\chi}\hat{J}_L}$ (see eq. (A.1)). Then by using the Poisson resummation formula $\sum_{n\in\mathbb{Z}}\exp(-\pi an^2+2\pi ibn) = \frac{1}{\sqrt{a}}\sum_{m\in\mathbb{Z}}\exp(-\pi(m-b)^2/a)$ we can show

$$\int(dC)^2\sum_{w,w'\in\mathbb{Z}}\exp\left[-\frac{\pi}{\alpha'\tau_2}\left(4C\bar{C}-2\bar{C}R\left(w'-w\tau\right)+2CR\left(w'-w\bar{\tau}\right)\right)+\right.$$
$$\left.+2\pi i\chi\hat{J}_R+2\pi i\bar{\chi}\hat{J}_L+2\pi i\tau\hat{N}_R-2\pi i\bar{\tau}\hat{N}_L\right]=$$
$$=\frac{\sqrt{(\alpha'\tau_2)^3}}{4R}\sum_{m,w\in\mathbb{Z}}\exp\left(-\pi\alpha'\tau_2M^2+2\pi\tau_1 i\left(\hat{N}_R-\hat{N}_L-nw\right)\right), \tag{2.12}$$

where M^2 is the mass spectrum [8] given by

$$\frac{\alpha'M^2}{2}=\frac{\alpha'}{2R^2}\left(n-qR\hat{J}\right)^2+\frac{R^2}{2\alpha'}\left(w-\frac{\alpha'}{R}\beta\hat{J}\right)^2+\hat{N}_R+\hat{N}_L-\hat{\gamma}\left(\hat{J}_R-\hat{J}_L\right),$$
$$\left(\hat{\gamma}\equiv\gamma-[\gamma]\,,\qquad \gamma\equiv qRw+\beta\alpha'\left(\frac{n}{R}-q\hat{J}\right)\right), \tag{2.13}$$

with the level matching constraint $\hat{N}_R-\hat{N}_L-nw=0$, where $\hat{J}$ is the total angular momentum $\hat{J}=\hat{J}_L+\hat{J}_R$, and $[\gamma]$ represents the largest integer which is less than γ. The integers $\hat{N}_{R,L}$ denote the number of state operators which include the zero point energy ($-1/2$ for NS-sector and 0 for R-sector).

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Let us see some interesting symmetry of the partition function $Z(R,q,\beta)$ [8]. From the mass spectrum it is easy to show the T-duality relation

$$Z(R,q,\beta) = Z\left(\frac{\alpha'}{R},\beta,q\right). \tag{2.14}$$

The interchange of q and β corresponds to that of metric $G_{\varphi y}$ and $B_{\varphi y}$, which is the essential part of T-duality transformation in the curved space. Furthermore one can see the periodicity of q and β from eq. (2.8)

$$Z(R,q,\beta) = Z\left(R,\ q+2n_1/R,\ \beta+2n_2R/\alpha'\right), \qquad (n_1,n_2\in\mathbb{Z}). \tag{2.15}$$

As we have seen, this model does not preserve any supersymmetry and thus it may be unstable. Therefore we would like to know whether the mass spectrum (2.13) includes tachyons in the NSNS sector. Then the answer is that it has tachyons if neither qR nor $\alpha'\beta/R$ is an integer [8]. If qR or $\alpha'\beta/R$ is an integer, then there is no tachyon for certain values of the radius [8]. In particular, for $(qR,\alpha'\beta/R)\in(2\mathbb{Z}+1,2\mathbb{Z})$, the model is identical to type-IIA(B) theory twisted by $(-1)^{F_S}\cdot\sigma_{1/2}$ [13] with radius $2R$, which is also equivalent to type-0B(A) theory twisted by $(-1)^{F_R}\cdot\sigma_{1/2}$ [12] with radius $1/R$. Here the operators F_S, F_R and $\sigma_{1/2}$ represent the spacetime fermion number, the world-sheet right-moving fermion number and the half-shift operator in the Y direction, respectively. If we further take the small radius limit $R\to 0$, we obtain ten-dimensional type-0 string theory in the T-dualized picture [12]. On the other hand, if we take the limit $R\to\infty$ with $\beta=0$, then the theory is identical to the ordinary ten-dimensional type-IIA(B) string theory. See also [32] for the similar relation between supersymmetric and non-supersymmetric heterotic string models.

In this way it is interesting to examine the small radius (or large radius in the T-dualized picture) limit of the Melvin backgrounds for various values of parameters q, β since it is expected to obtain some non-trivial decompactified ten-dimensional theories after T-duality. Inspired by this motivation let us consider the limit $R\to 0$ and $\beta\alpha'/R\to 0$ with the rational value $qR=k/N$, where k and N are coprime integers. Note that this limit is T-dual to $R\to\infty$ and $qR\to 0$ with $\beta\alpha'/R=k/N$ by using eq. (2.14). In the former picture with $\beta=0$ the redefined world-sheet fields (2.6) are twisted as follows

$$\begin{aligned}
X''(\sigma_1+2\pi,\sigma_2) &= e^{2\pi i\frac{k}{N}w}X''(\sigma_1,\sigma_2)\,,\\
S''^{r}_{R}(\sigma_1+2\pi,\sigma_2) &= e^{\pi i\frac{k}{N}w}S''^{r}_{R}(\sigma_1,\sigma_2)\,,\\
Y(\sigma_1+2\pi,\sigma_2) &= Y(\sigma_1,\sigma_2)+2\pi Rw\,.
\end{aligned} \tag{2.16}$$

Thus we can see that this string model is equivalent to a $\mathbb{Z}_N$ freely acting orbifold.[5] From this we speculate[6] that the limit $R\to 0$ corresponds to the orbifold $\mathbb{C}/\mathbb{Z}_N$. In order to show this exactly we investigate the one-loop partition function. The calculation in the

[5] For earlier discussions of the related Scherk-Schwarz compactification in string theory see for example [10, 11, 33, 34].

[6] In [6] the equivalence between the Melvin background and the orbifold $\mathbb{C}/\mathbb{Z}_N$ with H-flux was discussed for other case $q=\beta$ with finite radius.

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Green-Schwarz formalism is useful since in this formalism the flip of GSO-projection is automatically included due to spectral flow [8], which is crucial to determine that the theory is type 0 or type II.

The partition function (2.8) in this limit is given as follows[7] by using the identity (2.11) and the quasi-periodicity of theta-functions eq. (A.3)

$$\lim_{R\to 0} Z(R,q,\beta) = (2\pi)^{-7} V_7 R\left(\alpha'\right)^{-4} \int \frac{(d\tau)^2}{4(\tau_2)^5} \sum_{l,m=0}^{N-1} \sum_{\alpha,\beta\in\mathbb{Z}} \left(\lim_{R\to 0} e^{-\frac{\pi N^2 R^2}{\alpha' \tau_2}|\alpha-\beta\tau|^2}\right) \times$$
$$\times \frac{\left|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3 - (-1)^{k\alpha}\theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3 - (-1)^{k\beta}\theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3\right|^2}{4|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \quad (2.17)$$

where we have defined $\nu_{lm} = \frac{lk}{N} - \frac{mk}{N}\tau$, and integers $l,\ m,\ \alpha,\ \beta$ are given as $w' = N\alpha + l, w = N\beta + m$ $(l,m = 0,1,\ldots,N-1)$.

If k is an even integer, then the sign factors in front of the theta-functions are all plus and we obtain

$$Z(0,q,\beta) = V_1 V_7 \int \frac{(d\tau)^2}{4(\tau_2)} \left(4\pi^2\alpha'\tau_2\right)^{-4} \times$$
$$\times \sum_{l,m=0}^{N-1} \frac{\left|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3 - \theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3 - \theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3\right|^2}{4N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2}, \quad (2.18)$$

where the divergent factor V_1 is also given by $V_1 = \lim_{R\to 0} 2\pi\alpha'/NR$ and this corresponds to the volume of the non-compact direction. This value of radius α'/NR is consistent with that expected from the boundary condition (2.16) by T-duality. The sums over l and m in this expression (2.18) should be regarded as the $\mathbb{Z}_N$ projection $\frac{1}{N}\sum_{l=0}^{N-1} g^l$ $\ (g = \exp(2\pi i \frac{k}{N}\hat{J}))$ and the sum over twisted sectors, respectively. Therefore the model in this limit is identified with the orbifold $\mathbb{C}/\mathbb{Z}_N$ [35] in type-II string theory. As particular examples these orbifolds include those discussed in [20] $(k = N+1)$.

Next let us turn to the case where k is an odd integer. The result is

$$Z(0,q,\beta) = V_1' V_7 \int \frac{(d\tau)^2}{4(\tau_2)} \left(4\pi^2\alpha'\tau_2\right)^{-4} \sum_{l,m=0}^{N-1} \times$$
$$\times \frac{\left|\theta_3(\nu_{lm}|\tau)\theta_3(\tau)^3\right|^2 + \left|\theta_2(\nu_{lm}|\tau)\theta_2(\tau)^3\right|^2 + \left|\theta_4(\nu_{lm}|\tau)\theta_4(\tau)^3\right|^2}{2N|\eta(\tau)|^{18}\left|\theta_1(\nu_{lm}|\tau)\right|^2}, \quad (2.19)$$

where we have defined[8] $V_1' = V_1/2$. Thus this model is equivalent to the orbifold $\mathbb{C}/\mathbb{Z}_N$ in type-0 string theory with radius $\alpha'/2NR \to \infty$. The above results are summarized in figure 1.

[7] If lk/N and mk/N are both integers, then $\theta_1(\nu_{lm}|\tau)$ does vanish and the partition function will be divergent. This is due to the appearance of the zero modes of $(X'', \bar{X}'') \in \mathbb{R}^2$ and one should extract this divergence as the volume factor V_2.

[8] The extra factor 1/2 in comparison with the case of even k is understood if one notes that the "GSO-projection" in type 0 is the diagonal $\mathbb{Z}_2$ projection $(1+(-1)^{F_L+F_R})/2$, while in type-II theory it is give by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection $(1+(-1)^{F_L})(1+(-1)^{F_R})/4$.

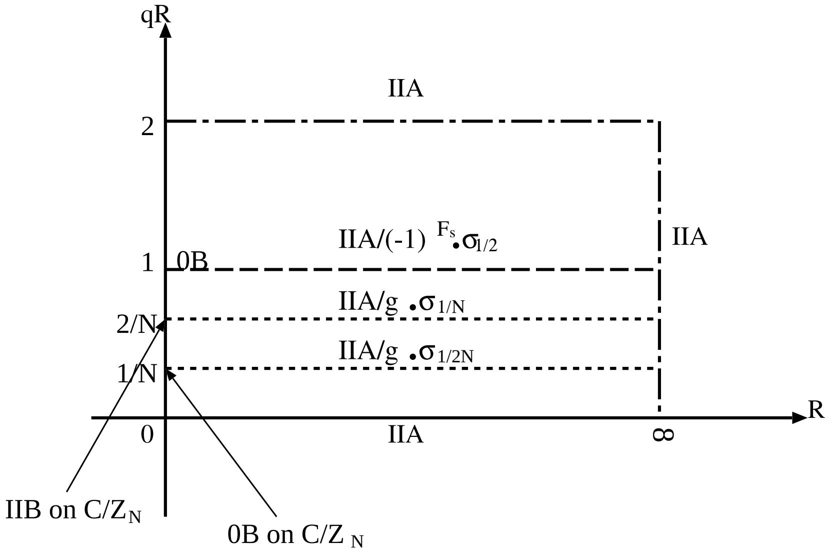

Figure 1: Moduli space of the string models in type-IIA Melvin backgrounds with $\beta = 0$.

This identification can also be seen from the mass spectrum (2.13). In the limit of $R \to 0$ we have the constraint $n = \frac{k}{N}\hat{J}$. This gives the correct $\mathbb{Z}_N$ orbifold projection. The shift of the energy $-\hat{\gamma}(\hat{J}_R - \hat{J}_L) = -(\widehat{km/N})(\hat{J}_R - \hat{J}_L)$ corresponds to the shift of modings in twisted sectors. If k is even, then $\hat{J}$ can be half integer and the NS-R and R-NS sector are allowed (remember that $\hat{J}$ is the total angular momentum.). On the other hand if k is odd, then those sectors are not allowed. This fact gives the difference between type-II and type-0 string theory. One can also read off the mass of lightest state for each twisted sectors. The result is given by for even k (type II)

$$\frac{\alpha' M^2}{2} = \begin{cases} -\hat{\mu} & \text{if } \ [\mu] \in \text{even}\,, \\ \hat{\mu} - 1 & \text{if } \ [\mu] \in \text{odd}\,, \end{cases} \tag{2.20}$$

and for odd k (type 0)

$$\frac{\alpha' M^2}{2} = \min\{\hat{\mu} - 1, -\hat{\mu}\}\,, \tag{2.21}$$

where we have defined $\mu = km/N$. The reason that the mass (2.20) depends on whether $[\mu]$ is even or odd is due to the "flip" of the sign of GSO projections by the spectral flow [8]. From the above results we can conclude that in the type-II orbifolds the tachyon appears in all twisted sectors while in type-0 orbifolds it does in untwisted sectors as well as in twisted sectors.

Now we would like to return to the detailed equivalence between the Melvin background with $qR = k/N, \beta = 0$ for the finite radius and the freely acting orbifold (2.16). By applying the Poisson resummation on α to the partition function (2.17), we can conclude that this special background is equivalent to the $\mathbb{Z}_N$ orbifold IIA(B)$/\sigma_{1/N} \cdot g$ ($g = \exp(2\pi i \frac{k}{N}\hat{J})$) with radius NR (for even k) or $\mathbb{Z}_{2N}$ orbifold IIA(B)$/\sigma_{1/2N} \cdot g$ with radius $2NR$ (for odd k). Here the operators $\sigma_{1/N}$ and $\sigma_{1/2N}$ mean $1/N$ and $1/2N$ shift along $\mathbf{S}^1$. The latter case is T-dual to the $\mathbb{Z}_N \times \mathbb{Z}_2$ orbifold $0\mathrm{B(A)}/\{(-1)^{F_R} \cdot \sigma_{1/2},\ \tilde{\sigma}_{1/N} \cdot g\}$ with radius $1/NR$, where the operator $\tilde{\sigma}_{1/N}$ is the T-dual to $\sigma_{1/N}$. For the special case $N = 1$ these results are reduced to the results in [12, 13] if we note the relation $g^N = (-1)^{F_S}$ for odd k.

Finally let us discuss the limit $R \to 0$ with irrational values of qR. The mass spectrum (2.13) shows the constraint $n - qR\hat{J} = 0$ again and this is satisfied iff $n = \hat{J} = 0$. Remarkably, we can not divide this system into a kind of a two-dimensional orbifold and one-dimensional non-compact space. This may also be regarded as a large-N limit of $\mathbb{Z}_N$ orbifold if we are reminded that any irrational number can be infinitely approximated by rational numbers. Note that taking this limit needs one extra dimension and the non-trivial background is given by the non-compact three-dimensional manifold. In any case we have to say that there are such unfamiliar non-compact string backgrounds, which interpolate the previous $\mathbb{Z}_N$ orbifolds.

A little analysis of the mass spectrum in this "irrational" model shows that tachyon fields appear in the sectors $w \neq 0$, and there exist tachyon fields whose mass is given by $-1 < \alpha' M^2/2 < -1 + \epsilon$ for any infinitesimal ϵ.

3. ALE space from higher-dimensional Melvin backgrounds

The closed string backgrounds we have discussed above do not preserve any supersymmetry in general. However, it is possible to realize the supersymmetry in higher-dimensional generalizations of the Melvin background as we will see below.

Let us consider a background of the form $\mathbb{M}_5 \times \mathbb{R}^{1,4}$, where $\mathbb{M}_5$ is a fibration of $\mathbf{S}^1 \ni Y$ over $\mathbb{R}^2 \times \mathbb{R}^2 \ni (X^1, \bar{X}^1) \times (X^2, \bar{X}^2)$. At the sigma model level this is possible if one assumes the higher-dimensional generalizations of (2.3):

$$\begin{aligned} S = \frac{1}{\pi\alpha'} \int (d\sigma)^2 \, \big[& (\partial + i\beta_1 V + iq_1 \partial Y) X^1 \left(\bar{\partial} - i\beta_1 \bar{V} - iq_1 \bar{\partial} Y\right) \bar{X}^1 + \\ & + (\partial + i\beta_2 V + iq_2 \partial Y) X^2 \left(\bar{\partial} - i\beta_2 \bar{V} - iq_2 \bar{\partial} Y\right) \bar{X}^2 + \\ & + V\bar{V} - \bar{V}\partial Y + V \bar{\partial} Y \big] \,, \end{aligned} \tag{3.1}$$

where we have omitted the fermion terms. The explicit metric of this model is given in the appendix B. The special case $q_1 = q_2, \beta_1 = \beta_2 = 0$ can be regarded as the $9 - 11$ flip of the supersymmetric F5-brane [16, 26]. The free field representation is again possible almost in the same way as before. If we turn to the Green-Schwarz formulation, the four of eight (light-cone gauge) spinor fields do not suffer from the phase factor when σ_1 is shifted by 2π if $q_1 = q_2, \beta_1 = \beta_2$ or $q_1 = -q_2, \beta_1 = -\beta_2$. Therefore we can conclude that in these cases half of thirty two supersymmetries are preserved.[9] From the supergravity viewpoint, we can see this as follows. For simplicity, let us set $\beta_1 = \beta_2 = 0$. Then if we go around the circle $\mathbf{S}^1$, the spinor fields obtain the phase $e^{i\pi(\pm q_1 \pm q_2)R}$. Thus if $q_1 = q_2$ or $q_1 = -q_2$, there are sixteen Killing spinors. We would also like to mention that similar arguments can also be generalized into seven or nine-dimensional background

[9] The mechanism of preserving supersymmetry in our model is closely related to the compactified model discussed in [34].

$\mathbb{M}_7$ and $\mathbb{M}_9$ which are fibrations of $S^1 \ni Y$ over $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The "9-11" flip of these will include the supersymmetric F3-brane, F1-brane (see also [26, 28]).

Another proof of the existence of the supersymmetry is to check the vanishing of the partition function, which is equivalent to the Bose-Fermi degeneracy. In the path-integral formulation of Green-Schwarz string one can compute this as before. The result is given by

$$\begin{aligned} Z(R, q_1, q_2, \beta_1, \beta_2) &= (2\pi)^{-5} V_5 R \left(\alpha'\right)^{-4} \int \frac{(d\tau)^2}{(\tau_2)^5} \int (dC)^2 \times \\ &\times \sum_{w, w' \in \mathbb{Z}} \frac{\left|\theta_1\left(\frac{\chi_1+\chi_2}{2}|\tau\right)\theta_1\left(\frac{\chi_1-\chi_2}{2}|\tau\right)\right|^4}{|\eta(\tau)|^{12}|\theta_1(\chi_1|\tau)\theta_1(\chi_2|\tau)|^2} \times \\ &\times \exp\left[-\frac{\pi}{\alpha'\tau_2}\left(4C\bar{C} - 2\bar{C}R\left(w' - w\tau\right) + 2CR\left(w' - w\bar{\tau}\right)\right)\right], \end{aligned} \tag{3.2}$$

where we have defined

$$\chi_1 = 2\beta_1 C + q_1 R\left(w' - \tau w\right), \qquad \chi_2 = 2\beta_2 C + q_2 R\left(w' - \tau w\right). \tag{3.3}$$

Thus it is easy to see the vanishing of $Z(R, q_1, q_2, \beta_1, \beta_2)$ if $\chi_1 = \chi_2$ or $\chi_1 = -\chi_2$, which is equivalent to $q_1 = q_2, \beta_1 = \beta_2$ or $q_1 = -q_2, \beta_1 = -\beta_2$.

If one wants the partition function in NS-R formalism, then one has only to note the Jacobi identity eq. (A.4) again

$$\begin{aligned} &2\theta_1\left(\frac{\chi_1}{2} + \frac{\chi_2}{2}\middle|\,\tau\right)^2 \theta_1\left(\frac{\chi_1}{2} - \frac{\chi_2}{2}\middle|\,\tau\right)^2 = \\ &\quad = \theta_3(\chi_1|\tau)\theta_3(\chi_2|\tau)\theta_3(0|\tau)^2 - \theta_2(\chi_1|\tau)\theta_2(\chi_2|\tau)\theta_2(0|\tau)^2 - \\ &\quad - \theta_4(\chi_1|\tau)\theta_4(\chi_2|\tau)\theta_4(0|\tau)^2. \end{aligned} \tag{3.4}$$

Next we can obtain the mass spectrum from (3.2) by the Poisson resummation formula as follows

$$\begin{aligned} \frac{\alpha' M^2}{2} &= \frac{\alpha'}{2R^2}\left(n - q_1 R\hat{J}_1 - q_2 R\hat{J}_2\right)^2 + \frac{R^2}{2\alpha'}\left(w - \frac{\alpha'}{R}\beta_1\hat{J}_1 - \frac{\alpha'}{R}\beta_2\hat{J}_2\right)^2 + \\ &+ \hat{N}_R + \hat{N}_L - \sum_{i=1}^{2}\hat{\gamma}_i\left(\hat{J}_{Ri} - \hat{J}_{Li}\right), \\ &\left(\hat{\gamma}_i \equiv \gamma_i - [\gamma_i], \qquad \gamma_i \equiv q_i R w + \beta_i \alpha'\left(\frac{n}{R} - q_1\hat{J}_1 - q_2\hat{J}_2\right),\right) \end{aligned} \tag{3.5}$$

with the level matching constraint $\hat{N}_R - \hat{N}_L - nw = 0$. The U(1) charges $\hat{J}_1$ and $\hat{J}_2$ are angular momentum operators for X^1 and X^2 directions, respectively.

From the above expression we can find the T-duality symmetry $R \leftrightarrow \alpha'/R$ and $q_i \leftrightarrow \beta_i$ if $q_1\beta_2 = q_2\beta_1$. The supersymmetric model satisfies this condition. We can also prove the periodicity like eq. (2.15).

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In the supersymmetric case we have found the Bose-Fermi degeneracy. Therefore this system should have no tachyons. Let us show this explicitly using the mass spectrum (3.5). We can assume $0 < \gamma_1 = \gamma_2 < 1$ without any loss of generality since there are no flip of GSO-projection if $\gamma_1 = \gamma_2$. Taking the GSO-projection into consideration, we obtain the inequalities $\hat{J}_{1R} + \hat{J}_{2R} \leq \hat{N}_R$ and $-\hat{N}_L \leq \hat{J}_{1L} + \hat{J}_{2L}$. Then it is easy to see in the NSNS sector

$$\frac{\alpha' M^2}{2} = \left(\hat{N}_R - \gamma\hat{J}_{1R} - \gamma\hat{J}_{2R}\right) + \left(\hat{N}_L + \gamma_1\hat{J}_{1L} + \gamma_2\hat{J}_{2L}\right) + \frac{\alpha'}{4}\left(P_R^2 + P_L^2\right) \geq 0\,. \qquad (3.6)$$

Now we would like to discuss the relation between the above models and orbifolds. We consider both supersymmetric and non-supersymmetric cases. Let us take the limit $R \to 0$ with $\beta_1\alpha'/R \to 0$ and $\beta_2\alpha'/R \to 0$, and assume that $q_1 R$ and $q_2 R$ are fractional. We can write[10] them as $q_1 R = k_1/N$ and $q_2 R = k_2/N$. Then the partition function in this limit becomes as in the previous calculations

$$\begin{aligned}
&\lim_{R\to 0} Z(R, q_1, q_2, \beta_1, \beta_2) = \\
&\quad = (2\pi)^{-5} V_5 R\left(\alpha'\right)^{-3} \int \frac{(d\tau)^2}{16(\tau_2)^4}\left(\lim_{R\to 0}\sum_{\alpha,\beta\in\mathbb{Z}} e^{-\frac{\pi N^2 R^2}{\alpha'\tau_2}|\alpha-\beta\tau|^2}\right)|\eta(\tau)|^{-12} \\
&\qquad \times \sum_{l,m=0}^{N-1}\Big|\theta_3\left(\nu^1_{l,m}|\tau\right)\theta_3\left(\nu^2_{l,m}|\tau\right)\theta_3(\tau)^2 - \\
&\qquad\qquad - (-1)^{(k_1+k_2)\alpha}\theta_2\left(\nu^1_{l,m}|\tau\right)\theta_2\left(\nu^2_{l,m}|\tau\right)\theta_2(\tau)^2 - \\
&\qquad\qquad - (-1)^{(k_1+k_2)\beta}\theta_4\left(\nu^1_{l,m}|\tau\right)\theta_4\left(\nu^2_{l,m}|\tau\right)\theta_4(\tau)^2\Big|^2 \cdot \\
&\qquad\qquad \cdot\left|\theta_1\left(\nu^1_{l,m}|\tau\right)\theta_1\left(\nu^2_{l,m}|\tau\right)\right|^{-2}, \qquad (3.7)
\end{aligned}$$

where we have defined $\nu^1_{l,m} = \frac{k_1}{N}(l - m\tau)$ and $\nu^2_{l,m} = \frac{k_2}{N}(l - m\tau)$.

If $k_1 + k_2$ is even, then we get the result

$$\begin{aligned}
&Z(0, q_1, q_2, \beta_1, \beta_2) = V_1 V_5 \int \frac{(d\tau)^2}{4\tau_2}\left(4\pi^2\alpha'\tau_2\right)^{-3}\sum_{l,m=0}^{N-1} \times \qquad (3.8) \\
&\times \frac{|\theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2 - \theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2 - \theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2|^2}{4N\ |\eta(\tau)|^{12}|\theta_1(\nu^1_{l,m}|\tau)\theta_1(\nu^2_{l,m}|\tau)|^2}\,.
\end{aligned}$$

Thus we have the abelian non-compact four-dimensional orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ in type-II string theory. These include both the supersymmetric and non-supersymmetric orbifolds. The former correspond to the values $k_1 = \pm k_2$ and it is easy to see that for fixed N the partition functions (3.8) for each k_1, k_2 give the same value. This represents the A_{N-1}-type ALE space (for a review see [23]) in the orbifold limit.

[10]Here we assume that there is no positive integer other than one which divides all of the three integers N, k_1 and k_2.

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The other orbifolds are all non supersymmetric and the tachyon can appear only in the twisted sectors. These include the examples discussed in [20], where the specific non-supersymmetric orbifolds are argued to decay into ALE spaces. Our results show that both such non-supersymmetric orbifolds and supersymmetric ALE orbifolds (including the type-II string in flat space $N = 1$) are connected in the moduli space of solvable superstring models if we compactify one direction. In other words, these two different kind of orbifolds are all obtained from each other by marginal deformations. We will discuss the decay of such non-supersymmetric orbifolds in the last section.

Next we turn to the case where $k_1 + k_2$ is odd. The partition function is given by

$$Z(0, q_1, q_2, \beta_1, \beta_2) = V_1' V_7 \int \frac{(d\tau)^2}{4\tau_2} (4\pi^2 \alpha' \tau_2)^{-3} \sum_{l,m=0}^{N-1} \times \tag{3.9}$$

$$\times \frac{|\theta_3(\nu^1_{l,m}|\tau)\theta_3(\nu^2_{l,m}|\tau)\theta_3(\tau)^2|^2 + |\theta_2(\nu^1_{l,m}|\tau)\theta_2(\nu^2_{l,m}|\tau)\theta_2(\tau)^2|^2 + |\theta_4(\nu^1_{l,m}|\tau)\theta_4(\nu^2_{l,m}|\tau)\theta_4(\tau)^2|^2}{2N|\eta(\tau)|^{12}|\theta_1(\nu^1_{l,m}|\tau)\theta_1(\nu^2_{l,m}|\tau)|^2}.$$

This explicitly shows that the systems now considered are equivalent to the non-compact four-dimensional orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ in type-0 string theory.[11] The above result shows that orbifolds in type-0 theory are connected to those in type-II theory. In particular, this shows that various orbifolds in type-0 theory can be regarded as non-supersymmetric backgrounds in type-II string theory.

Then let us discuss the tachyons in these orbifolds. The mass of the lightest state is given by for even $k_1 + k_2$ (type II)

$$\frac{\alpha' M^2}{2} = \begin{cases} -|\hat{\mu_1} - \hat{\mu_2}| & \text{if } ([\mu_1], [\mu_2]) \in (\text{even,even}) \text{ or } (\text{odd,odd})\,, \\ \hat{\mu_1} + \hat{\mu_2} - 1 & \text{if } ([\mu_1], [\mu_2]) \in (\text{even,odd}) \text{ or } (\text{odd,even})\,, \end{cases} \tag{3.10}$$

and for odd $k_1 + k_2$ (type 0)

$$\frac{\alpha' M^2}{2} = \min\{\hat{\mu_1} + \hat{\mu_2} - 1, -|\hat{\mu_1} - \hat{\mu_2}|\}\,, \tag{3.11}$$

where we have defined $\mu_1 = k_1 m/N$ and $\mu_2 = k_2 m/N$. These results show that in both orbifolds some of the twisted sectors will contain tachyon and in type-0 orbifolds tachyon also appears in the untwisted sector. The type-II string theory on ALE orbifolds $k_1 = \pm k_2$ are only examples of tachyon less orbifolds.

Finally, if we turn to the small radius limit with irrational values of $q_1 R$ and $q_2 R$, we obtain the "irrationally orbifolded" non-compact space (or a kind of a "large-N limit of the orbifolds $\mathbb{C}^2/\mathbb{Z}_\infty$") as in the original Melvin background. For specific values $q_1 = \pm q_2$ this background preserves the half of thirty two supersymmetries and is connected to ALE spaces.

It will be also interesting to consider the brane picture of the general supersymmetric backgrounds with $\beta_{1,2} \neq 0$, which includes the previous orbifolds as special examples by T-duality. These models have the non-trivial H-flux and dilaton gradient. We left its relation to NS5-branes as a future problem.

[11] Such orbifolds are considered in the context of D-branes in [36].

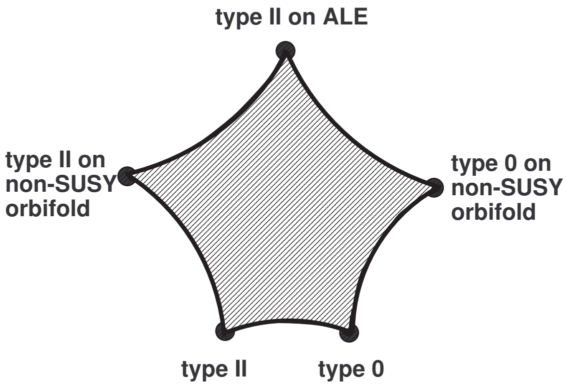

Figure 2: Moduli space of the string models in the higher-dimensional Melvin background.

In this way we have shown that the various orbifolds both non supersymmetric and supersymmetric are included in the moduli space of the higher-dimensional Melvin background (see figure 2).

4. Conclusions and discussions

In this paper we have discussed the conformal field theory of the Melvin background and its higher-dimensional generalization. In the latter one we found new examples of solvable supersymmetric string models on non-compact five-dimensional manifolds (see eq. (B.1)) with H-flux. We have shown that various abelian orbifolds in type-II and type-0 string theory can be obtained in the small radius (or large radius in the T-dualized picture) limit of the Melvin background with the rational values of the magnetic parameters. Even though the examples we have examined are two and four-dimensional orbifolds $\mathbb{C}/\mathbb{Z}_N,\ \mathbb{C}^2/\mathbb{Z}_N$, our results will be easily generalized into much higher-dimensional orbifolds $\mathbb{C}^n/\mathbb{Z}_N,\ n=3,4$. Other abelian orbifolds such as $\mathbb{C}^n/\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\cdots\times\mathbb{Z}_{N_K}$ can also be obtained if we replace $\mathbf{S}^1$ with K-dimensional tori. Non-abelian orbifolds (e.g. D,E type) will deserve future study. The T-dual counterparts of these examples are also interesting since the presence of the H-flux is essential in such models. They seem to have some relevance to NS5-branes.

One of the important lessons which can be obtained from the above arguments is that we can view both non-supersymmetric and supersymmetric orbifolds as marginal deformations of flat type-II or type-0 string theory. This can be seen as a new example of the duality relation between stable supersymmetric string theories and unstable string theories which include tachyons. Thus it is intriguing to explore the application to the decay of unstable spacetime or closed string tachyon condensation.

Let us give a very intuitive idea about the decay processes of these unstable backgrounds. Since we discuss only exactly marginal deformations, we have no tree level potential. Instead the leading quantity which determines the stability will be the one-loop amplitude (or cosmological constant). Even though this value is IR divergent in the general parameter region of the Melvin background, this divergence can be estimated by the tachyon mass whose absolute value is largest. Thus we can expect that the unstable tachyonic background will decay into another background which is less tachyonic (the absolute value of the tachyon mass square is smaller) and eventually into the tachyon less background. Let us take the example of the orbifold $\mathbb{C}/\mathbb{Z}_N$. If we are reminded of the moduli space (see figure 1 or figure 2) of Melvin background, we find many decay routes. The most straightforward way is to decay into the ordinary type-II string vacuum by the spontaneous (de)compactification of Y direction. Another possibility is the shift of the magnetic parameter. However the infinitesimal shift will make the value of qR irrational. Since such a background is more tachyonic as we saw in the last of section 2, this decay route is not preferable. On the other hand, the decay mode from $\mathbb{C}/\mathbb{Z}_{2l+1}$ into $\mathbb{C}/\mathbb{Z}_{2l-1}$ discussed in [20] is more favorable because the absolute value of tachyon mass decreases. Note that the above arguments are not involved in the real closed string tachyon condensation by relevant perturbations as contrasted with [20]. This viewpoint may be supported by the observed absence of tree level tachyon potential [14, 37].

Another important lesson is that the limits of the Melvin backgrounds depend very sensitively on whether the value of the magnetic flux is rational or irrational. In the latter case one can find unfamiliar supersymmetric string backgrounds, which can be regarded as a "large-N orbifold". The existence of two remarkably different kinds of limits can also be seen in the D-brane spectrums on such backgrounds as we will discuss in our forthcoming paper [29].

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A. Identities of theta functions

Here we summarize the formulae of θ-functions. First define the following θ-functions:

$$\begin{aligned}
\eta(\tau) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,,\\
\theta_1(\nu|\tau) &= 2q^{1/8}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1-e^{2i\pi\nu}q^n\right)\left(1-e^{-2i\pi\nu}q^n\right),\\
\theta_2(\nu|\tau) &= 2q^{1/8}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1+e^{2i\pi\nu}q^n\right)\left(1+e^{-2i\pi\nu}q^n\right),\\
\theta_3(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1+e^{2i\pi\nu}q^{n-\frac{1}{2}}\right)\left(1+e^{-2i\pi\nu}q^{n-\frac{1}{2}}\right),\\
\theta_4(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1-e^{2i\pi\nu}q^{n-\frac{1}{2}}\right)\left(1-e^{-2i\pi\nu}q^{n-\frac{1}{2}}\right),
\end{aligned}\tag{A.1}$$

where we have defined $q=e^{2i\pi\tau}$.

Next we show the modular properties as follows

$$
\begin{aligned}
\eta(\tau) &= (-i\tau)^{-1/2}\eta\left(-\frac{1}{\tau}\right), \qquad \theta_1(\nu|\tau) = i(-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_1\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right),\\
\theta_2(\nu|\tau) &= (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_4\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right), \qquad \theta_3(\nu|\tau) = (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_3\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right),\\
\theta_4(\nu|\tau) &= (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_2\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right).
\end{aligned} \tag{A.2}
$$

Their quasi periodicity is also given by

$$
\begin{aligned}
\theta_1(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_1(\nu|\tau)\,,\\
\theta_2(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_2(\nu|\tau)\,,\\
\theta_3(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_3(\nu|\tau)\,,\\
\theta_4(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_4(\nu|\tau)\,.
\end{aligned} \tag{A.3}
$$

It is also useful to note the Jacobi's identity

$$
\prod_{a=1}^4\theta_3(\nu_a|\tau)-\prod_{a=1}^4\theta_2(\nu_a|\tau)-\prod_{a=1}^4\theta_4(\nu_a|\tau)+\prod_{a=1}^4\theta_1(\nu_a|\tau)=2\prod_{a=1}^4\theta_1\left(\nu_a'|\tau\right), \tag{A.4}
$$

where we have defined

$$
\begin{aligned}
2\nu_1' &= \nu_1+\nu_2+\nu_3+\nu_4\,, \qquad 2\nu_2' = \nu_1+\nu_2-\nu_3-\nu_4\,,\\
2\nu_3' &= \nu_1-\nu_2+\nu_3-\nu_4\,, \qquad 2\nu_4' = \nu_1-\nu_2-\nu_3+\nu_4\,.
\end{aligned} \tag{A.5}
$$

B. Explicit metric of the higher-dimensional model

Let us define the polar coordinates as $X^1=\rho e^{i\varphi}$, $X^2=re^{i\theta}$. The metric (in string frame) is given by

$$
\begin{aligned}
(ds)^2 &= d\rho^2+dr^2+\frac{1+\beta_2^2r^2}{F(r,\rho)}\rho^2d\varphi^2-2\frac{\beta_1\beta_2r^2\rho^2}{F(r,\rho)}d\varphi d\theta+\frac{1+\beta_1^2\rho^2}{F(r,\rho)}r^2d\theta^2+\\
&\quad+\frac{G(r,\rho)}{F(r,\rho)}dy^2+2\frac{q_1\rho^2\left(1+\beta_2^2r^2\right)-\beta_1\beta_2q_2\rho^2r^2}{F(r,\rho)}d\varphi dy+\\
&\quad+2\frac{q_2r^2\left(1+\beta_1^2\rho^2\right)-\beta_1\beta_2q_1\rho^2r^2}{F(r,\rho)}d\theta dy\\
&= \frac{G(r,\rho)}{F(r,\rho)}(dy+A_\varphi d\varphi+A_\theta d\theta)^2-\frac{2r^2\rho^2}{G(r,\rho)F(r,\rho)}\left(\beta_1\beta_2+q_1q_2F(r,\rho)\right)d\varphi d\theta+\\
&\quad+d\rho^2+\frac{\rho^2}{G(r,\rho)F(r,\rho)}\left(1+\beta_2^2r^2+q_2^2r^2+q_2^2\beta_2^2r^4+\beta_1^2q_2^2\rho^2r^2\right)d\varphi^2+\\
&\quad+dr^2+\frac{r^2}{G(r,\rho)F(r,\rho)}\left(1+\beta_1^2\rho^2+q_1^2\rho^2+q_1^2\beta_1^2\rho^4+\beta_2^2q_1^2\rho^2r^2\right)d\theta^2\,,
\end{aligned} \tag{B.1}
$$

where we have defined

$$\begin{aligned} F(r,\rho) &\equiv 1+\beta_1^2\rho^2+\beta_2^2 r^2\,, \qquad G(r,\rho)\equiv 1+(\beta_1 q_2-\beta_2 q_1)^2\rho^2 r^2+q_1^2\rho^2+q_2^2 r^2\,,\\ A_\varphi &= \frac{q_1\rho^2+\beta_2(q_1\beta_2-q_2\beta_1)\rho^2 r^2}{G(r,\rho)}\,, \qquad A_\theta=\frac{q_2 r^2+\beta_1(q_2\beta_1-q_1\beta_2)\rho^2 r^2}{G(r,\rho)}\,. \end{aligned} \tag{B.2}$$

The B-field and the dilaton is

$$B_\varphi=-\frac{\beta_1\rho^2}{F(r,\rho)}\,, \qquad B_\theta=-\frac{\beta_2 r^2}{F(r,\rho)}\,, \qquad e^{2(\phi-\phi_0)}=\frac{1}{F(r,\rho)}\,. \tag{B.3}$$

One can check that the curvature of the above metric is not singular.

It is not so difficult to show that these satisfy the equations of motion in supergravity

$$\begin{aligned} R_{\mu\nu}+\nabla_\mu\nabla_\nu\phi-\frac{1}{4}H_{\mu\alpha\beta}H_\nu^{\alpha\beta} &= 0\,,\\ -\frac{1}{2}\nabla^2\phi+\nabla_\mu\phi\nabla^\mu\phi-\frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} &= 0\,,\\ \nabla_\rho H^\rho_{\mu\nu}-2(\nabla_\rho\phi)H^\rho_{\mu\nu} &= 0\,. \end{aligned} \tag{B.4}$$

It is also worth noting that we can relate $B_{\varphi,\theta}$ to $A_{\varphi,\theta}$ by T-duality: $R\leftrightarrow\alpha'/R$ and $q_i\leftrightarrow\beta_i$ if $(\beta_1 q_2-\beta_2 q_1)=0$. This includes the supersymmetric cases $q_1=\pm q_2$, $\beta_1=\pm\beta_2$.

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